

Proofonomicon

A Reference of the veriT Proof Format

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1 Introduction

This document is a reference manual for the format used by the SMT solver veriT to print proofs of the unsatisfiability of the input problem. It is part of the veriT repository and represents the format as generated by the corresponding source code. Currently the introduction gives a general overview of the proof format and lacks a complete formal definition of the calculus. Section 5 gives a complete lists of proof rules currently used by veriT. This section is probably the one of greatest utility to the reader.

1.1 Overview

veriT [4] is a CDCL(T)-based satisfiability modulo theories solver. It uses the SMT-LIB language as input and output language and also utilizes the many-sorted classical first-order logic defined by this language. If requested by the user, veriT outputs a proof if it can deduce that the input problem is unsatisfiable. In proof production mode, veriT supports the theory of uninterpreted functions, the theory of linear integer and real arithmetic, and quantifiers.

Similar to the proofs generated by Z3, veriT's proofs are based on the SMT-LIB language, but are otherwise different. Proofs are not terms, but a list of indexed steps. Steps without references are tautologies and assumptions. The last step is always the deduction of the empty clause. Furthermore, steps can be marked as subproofs, which are used for local assumptions and to reason about bound variables. To shorten the proof length, veriT uses term sharing. This is implemented using the standard SMT-LIB name annotation mechanism. Major differences to the proof format used by Z3 are the fine-grained steps for Skolemization and the presence of steps for the manipulations of bound variables.

In addition to this reference, the proof format used by veriT has been discussed in publications which provide valuable background information: the fundamental ideas behind the proof format have been published in [3]; proposed rules for quantifier instantiation can be found in [5]; and the proof rules to express reasoning typically used for processing, such as Skolemization, renaming of variables, and other manipulations of bound variables have been published in [1]. A complete reconstruction of the proofs generated by veriT in Isabelle/HOL has recently been reported in [6]. Parts of this manual have been taken from this publication.

2 Notation

veriT uses the SMT-LIB language [2] as both its input and output language. Hence, this document builds on the concept introduced there. This includes the concrete syntax and the multi-sorted first-order logic.

The notation used throughout this manual follows the notation of the SMT-LIB standard. To simplify the notation we will omit the sort of terms when possible. The available sorts depend on the selected SMT-LIB theory and can also be extended by the user, but a distinguished **Bool** sort is always available. We use the symbols x , y , z for variables,

f, g, h for functions, and P, Q for predicates, i.e., functions with result sort **Bool**. The symbols r, s, t, u stand for terms. The symbols φ, ψ denote formulas, i.e., terms of sort **Bool**. We use σ to denote substitutions and $t\sigma$ to denote the application of the substitution on the term t . To denote the substitution which maps x to t we write $[t/x]$. We use $=$ to denote syntactic equality and \simeq to denote the sorted equality predicate. We also use the notion of complementary literals very liberally: $\varphi = \bar{\psi}$ holds if the terms obtained after removing all leading negations from φ and $\bar{\psi}$ are syntactically equal and the number of leading negations is even for φ and odd for $\bar{\psi}$, or vice versa.

A proof generated by veriT is a list of steps. A step consists of an index $i \in \mathbb{N}$, a formula φ , a rule name R taken from a set of possible rules, a possibly empty set of premises $\{p_1, \dots, p_n\}$ with $p_i \in \mathbb{N}$, a rule-dependent and possibly empty list of arguments $[a_1, \dots, a_m]$, and a context Γ . The arguments a_i are either terms or tuples (x_i, t_i) where x_i is a variable and t_i is a term. The interpretation of the arguments is rule specific. The context is a possible empty list $[c_1, \dots, c_l]$, where c_i stands for either a variable or a variable-term tuple (x_i, t_i) . A context denotes a substitution as described in section 3. Every proof ends with a step with the empty clause as the step term and empty context. The list of premises only references earlier steps, such that the proof forms a directed acyclic graph. In section 5 we provide an overview of all proof rules used by veriT.

To mimic the actual proof text generated by veriT we will use the following notation to denote a step:

$$c_1, \dots, c_l \triangleright i. \varphi \text{ (rule; } p_1, \dots, p_n; a_1, \dots, a_m)$$

If an element of the context c_i is of the form (x_i, t_i) , we will write $x_i \mapsto t_i$. If an element of the arguments a_i is of this form we will write $x_i := t_i$. Furthermore, the proofs can utilize Hilbert's choice operator ϵ . Choice acts like a binder. The term $\epsilon x.\varphi$ stands for a value v , such that $\varphi[v/x]$ is true if such a value exists. Any value is possible otherwise. Thorough this document, we use **i, j, k, l, m, n** for step indices.

3 Core Concepts of the Proof Format

Assumptions. The `assume` rule introduces a term as an assumption. The proof starts with a number of `assume` steps. Each step corresponds to an assertion. Additional assumptions can be introduced too. In this case each assumption must be discharged with an appropriate step. The only rule to do so is the `subproof` rule. From an assumption φ and a formula ψ proved by intermediate steps from φ , the `subproof` step deduces $\neg\varphi \vee \psi$ and discharges φ .

Tautologous rules and simple deduction. Most rules emitted by veriT introduce tautologies. One example is the `and_pos` rule: $\neg(\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n) \vee \varphi_i$. Other rules operate on only one premise. Those rules are primarily used to simplify Boolean connectives during preprocessing. For example, the `implies` rule removes an implication: From $\varphi_1 \implies \varphi_2$ it deduces $\neg\varphi_1 \vee \varphi_2$.

`sko_forall` works like `sko_ex`, except that the choice term is $\epsilon x. \neg \varphi$.

$$\begin{array}{lll}
x \mapsto (\epsilon x. \neg p(x)) \triangleright 1. & x \simeq \epsilon x. \neg p(x) & (\text{refl}) \\
x \mapsto (\epsilon x. \neg p(x)) \triangleright 2. & p(x) \simeq p(\epsilon x. \neg p(x)) & (\text{cong}; 1) \\
\triangleright 3. & (\forall x. p(x)) \simeq p(\epsilon x. \neg p(x)) & (\text{sko_forall}; 2) \\
\triangleright 4. & (\neg \forall x. p(x)) \simeq \neg p(\epsilon x. \neg p(x)) & (\text{cong}; 3)
\end{array}$$

Linear arithmetic. Proofs for linear arithmetic use a number of straightforward rules, such as `la_totality`: $t_1 \leq t_2 \vee t_2 \leq t_1$ and the main rule `la_generic`. The conclusion of an `la_generic` step is a tautology of the form $(\neg \varphi_1) \vee (\neg \varphi_2) \vee \dots \vee (\neg \varphi_n)$ where the φ_i are linear (in)equalities. Checking the validity of this formula amounts to checking the unsatisfiability of the system of linear equations $\varphi_1, \varphi_2, \dots, \varphi_n$.

Example 2. The following example is the proof generated by veriT for the unsatisfiability of $(x + y < 1) \vee (3 < x)$, $x \simeq 2$, and $0 \simeq y$.

$$\begin{array}{lll}
\triangleright 1. & (3 < x) \vee (x + y < 1) & (\text{assume}) \\
\triangleright 2. & x \simeq 2 & (\text{assume}) \\
\triangleright 3. & 0 \simeq y & (\text{assume}) \\
\triangleright 4. & \neg(3 < x) \vee \neg(x \simeq 2) & (\text{la_generic}; ;1.0, 1.0) \\
\triangleright 5. & \neg(3 < x) & (\text{resolution}; 2, 4) \\
\triangleright 6. & x + y < 1 & (\text{resolution}; 1, 5) \\
\triangleright 7. & \neg(x + y < 1) \vee \neg(x \simeq 2) \vee \neg(0 \simeq y) & (\text{la_generic}; ;1.0, 1.0, 1.0) \\
\triangleright 8. & \perp & (\text{resolution}; 7, 6, 2, 3)
\end{array}$$

Implicit transformations reordering of equalities. In addition to the explicit steps, veriT might reorder equalities, i.e. apply symmetry of the equality predicate, without generating steps. When this happens is somewhat restricted. Equalities are only reordered when the term below the equality change during proof search. One such case is the instantiation of universally quantified variables. If the variable that appears below an equality, then the equality might have an arbitrary order after the variable is instantiated.

4 The Concrete Syntax

The concrete text representation of the proofs generated by veriT is based on the SMT-LIB standard. Figure 1 shows an exemplary proof as printed by veriT lightly edited for readability. The format follows the SMT-LIB standard when possible.

Figure 2 shows the grammar of the proof text generated by veriT. It is based on the SMT-LIB grammar, as defined in the SMT-LIB standard version 2.6 Appendix B¹. The nonterminals `<symbol>`, `<function_def>`, `<sorted_var>`, and `<term>` are as defined in the standard. The `<proof_term>` is the recursive `<term>` nonterminal redefined with the additional production for the `choice` binder.

¹Available online at: <http://smtlib.cs.uiowa.edu/language.shtml>

```

1 (assume h1 (not (p a)))
2 (assume h2 (forall ((z1 U)) (forall ((z2 U)) (p z2))))
3 ...
4 (anchor :step t9 :args ((:= z2 vr4)))
5 (step t9.t1 (cl (= z2 vr4)) :rule refl)
6 (step t9.t2 (cl (= (p z2) (p vr4))) :rule cong :premises (t9.t1))
7 (step t9 (cl (= (forall ((z2 U)) (p z2)) (forall ((vr4 U)) (p vr4))))
8     :rule bind)
9 ...
10 (step t14 (cl (forall ((vr5 U)) (p vr5)))
11     :rule th_resolution :premises (t11 t12 t13))
12 (step t15 (cl (or (not (forall ((vr5 U)) (p vr5))) (p a)))
13     :rule forall_inst :args ((:= vr5 a)))
14 (step t16 (cl (not (forall ((vr5 U)) (p vr5))) (p a))
15     :rule or :premises (t15))
16 (step t17 (cl) :rule resolution :premises (t16 h1 t14))

```

Figure 1: Example proof output. Assumptions are introduced (line 1–2); a subproof renames bound variables (line 4–8); the proof finishes with instantiation and resolution steps (line 10–15)

Input problems in the SMT-LIB standard contain a list of *commands* that modify the internal state of the solver. In agreement with this approach veriT’s proofs are also formed by a list of commands. The `assume` command introduces a new assumption. While this command could also be expressed using the `step` command with a special rule, the special semantic of an assumption justifies the presence of a dedicated command: assumptions are neither tautological nor derived from premises. The `step` command, on the other hand, introduces a derived or tautological term. Both commands `assume` and `step` require an index as the first argument to later refer back to it. This index is an arbitrary SMT-LIB symbol. The only restriction is that it must be unique for each `assume` and `step` command. The second argument is the term introduced by the command. For a `step`, this term is always a clause. To express disjunctions in SMT-LIB the `or` operator is used. Unfortunately, this operator needs at least two arguments and cannot represent unary or empty clauses. To circumvent this we introduce a new `cl` operator. It corresponds the standard `or` function extended to one argument, where it is equal to the identity, and zero arguments, where it is equal to `false`. The `:premises` annotation denotes the premises and is skipped if they are none. If the rule carries arguments, the `:args` annotation is used to denote them.

The `anchor` and `define-fun` commands are used for subproofs and sharing, respectively. The `define-fun` command corresponds exactly to the `define-fun` command of the SMT-LIB language.

```

    <proof> ::= <proof_command>*
    <proof_command> ::= (assume <symbol> <proof_term> )
    | (step <symbol> <clause> :rule <symbol>
      <step_annotation> )
    | (anchor :step <symbol> )
    | (anchor :step <symbol> :args <proof_args> )
    | (define-fun <function_def> )
    <clause> ::= (cl <proof_term>* )
    <step_annotation> ::= :premises ( <symbol>+ )
    | :args <proof_args>
    | :premises ( <symbol>+ ) :args <proof_args>
    <proof_args> ::= ( <proof_arg>+ )
    <proof_arg> ::= <symbol> | ( <symbol> <proof_term> )
    <proof_term> ::= <term> extended with
    (choice ( <sorted_var>+ ) <proof_term> )

```

Figure 2: The proof grammar

4.1 Subproofs

As the name suggests, the **subproof** rule expresses subproofs. This is possible because its application is restricted: the assumption used as premise for the **subproof** step must be the assumption introduced last. Hence, the **assume**, **subproof** pairs are nested. The context is manipulated in the same way: if a step pops c_1, \dots, c_n from a context Γ , there is a earlier step which pushes precisely c_1, \dots, c_n onto the context. Since contexts can only be manipulated by push and pop, context manipulations are also nested.

Because of this nesting, veriT uses the concept of subproofs. A subproof is started right before an **assume** command or a command which pushes onto the context. We call this point the *anchor*. The subproof ends with the matching **subproof** command or command which pops from the context, respectively. The **:step** annotation of the anchor command is used to indicate the **step** command which will end the subproof. The term of this **step** command is the conclusion of the subproof. If the subproof uses a context, the **:args** annotation of the **anchor** command indicates the arguments added to the context for this subproof. In the example proof (Figure 1) a subproof starts on line four. It ends on line seven with the **bind** steps which finished the proof for the renaming of the bound variable **z2** to **vr4**.

A further restriction applies: only the conclusion of a subproof can be used as a premise outside of the subproof. Hence, a proof checking tool can remove the steps of the subproof from memory after checking it.

4.2 Sharing and Skolem Terms

The proof output generated by veriT is generally large. One reason for this is that veriT can store terms internally much more efficiently. By utilizing a perfect sharing data structure, every term is stored in memory precisely once. When printing the proof this compact storage is unfolded.

The user of veriT can optionally activate sharing² to print common subterms only once. This is realized using the standard naming mechanism of SMT-LIB. In the language of SMT-LIB it is possible to annotate every term t with a name n by writing `(! t :named n)` where n is a symbol. After a term is annotated with a name, the name can be used in place of the term. This is a purely syntactical replacement.

To simplify reconstruction veriT can optionally³ define Skolem constants as functions. If activated, this option adds a list of `define-fun` command to define shorthand 0-ary functions for the `(choice ...)` terms needed. Without this option, no `define-fun` commands are issued and the constants are inlined.

5 List of Proof Rules

The following lists all rules produced by veriT. When n -ary operators are in the form $t_1 \vee \dots \vee t_n$ this corresponds to the SMT-LIB string `(or t1 ... tn)`. Hence, we explicitly bracket to clarify the application. To differentiate between `or` from `cl`, we use $\dot{\vee}$ for the second operator. Nevertheless, since proof steps always start with `cl`, we write the literal of unit clauses directly. Furthermore, premises can be ordered arbitrarily and need not follow the order given in the rule definition.

1. assume

\triangleright i. ϕ (assume)

where ϕ is equivalent to a formula asserted in the input problem.

2. true

\triangleright i. \top (true)

3. false

\triangleright i. $\neg \perp$ (false)

4. not_not

²By using the command-line option `--proof-with-sharing`.

³By using the command-line option `--proof-define-skolems`.

- ▷ i. $\neg(\neg\neg\varphi)\dot{\vee}\varphi$ (not_not)
- 5. and_pos**
- ▷ i. $\neg(\varphi_1 \wedge \cdots \wedge \varphi_n)\dot{\vee}\varphi_i$ (and_pos)
with $1 \leq i \leq n$.
- 6. and_neg**
- ▷ i. $(\varphi_1 \wedge \cdots \wedge \varphi_n)\dot{\vee}(\neg\varphi_1)\dot{\vee}\dots\dot{\vee}(\neg\varphi_n)$ (and_neg)
- 7. or_pos**
- ▷ i. $\neg(\varphi_1 \vee \cdots \vee \varphi_n)\dot{\vee}\varphi_1\dot{\vee}\dots\dot{\vee}\varphi_n$ (or_pos)
- 8. or_neg**
- ▷ i. $(\varphi_1 \vee \cdots \vee \varphi_n)\dot{\vee}(\neg\varphi_i)$ (or_neg)
with $1 \leq i \leq n$.
- 9. xor_pos1**
- ▷ i. $\neg(\varphi_1 \text{ xor } \varphi_2)\dot{\vee}\varphi_1\dot{\vee}\varphi_2$ (xor_pos1)
- 10. xor_pos2**
- ▷ i. $\neg(\varphi_1 \text{ xor } \varphi_2)\dot{\vee}(\neg\varphi_1)\dot{\vee}(\neg\varphi_2)$ (xor_pos2)
- 11. xor_neg1**
- ▷ i. $(\varphi_1 \text{ xor } \varphi_2)\dot{\vee}\varphi_1\dot{\vee}(\neg\varphi_2)$ (xor_neg1)
- 12. xor_neg2**
- ▷ i. $(\varphi_1 \text{ xor } \varphi_2)\dot{\vee}(\neg\varphi_1)\dot{\vee}\varphi_2$ (xor_neg2)
- 13. implies_pos**
- ▷ i. $\neg(\varphi_1 \rightarrow \varphi_2)\dot{\vee}(\neg\varphi_1)\dot{\vee}\varphi_2$ (implies_pos)

14. implies_neg1

▷ i. $(\varphi_1 \rightarrow \varphi_2) \dot{\vee} \varphi_1$ (implies_neg1)

15. implies_neg2

▷ i. $(\varphi_1 \rightarrow \varphi_2) \dot{\vee} (\neg \varphi_2)$ (implies_neg2)

16. equiv_pos1

▷ i. $\neg(\varphi_1 \leftrightarrow \varphi_2) \dot{\vee} \varphi_1 \dot{\vee} (\neg \varphi_2)$ (equiv_pos1)

17. equiv_pos2

▷ i. $\neg(\varphi_1 \leftrightarrow \varphi_2) \dot{\vee} (\neg \varphi_1) \dot{\vee} \varphi_2$ (equiv_pos2)

18. equiv_neg1

▷ i. $\varphi_1 \leftrightarrow \varphi_2 \dot{\vee} (\neg \varphi_1) \dot{\vee} (\neg \varphi_2)$ (equiv_neg1)

19. equiv_neg2

▷ i. $\varphi_1 \leftrightarrow \varphi_2 \dot{\vee} \varphi_1 \dot{\vee} \varphi_2$ (equiv_neg2)

20. ite_pos1

▷ i. $\neg(\text{ite } \varphi_1 \varphi_2 \varphi_3) \dot{\vee} \varphi_1 \dot{\vee} \varphi_3$ (ite_pos1)

21. ite_pos2

▷ i. $\neg(\text{ite } \varphi_1 \varphi_2 \varphi_3) \dot{\vee} (\neg \varphi_1) \dot{\vee} \varphi_2$ (ite_pos2)

22. ite_neg1

▷ i. $\text{ite } \varphi_1 \varphi_2 \varphi_3 \dot{\vee} \varphi_1 \dot{\vee} (\neg \varphi_3)$ (ite_neg1)

23. ite_neg2

▷ i. $\text{ite } \varphi_1 \varphi_2 \varphi_3 \dot{\vee} (\neg \varphi_1) \dot{\vee} (\neg \varphi_2)$ (ite_neg2)

24. eq_reflexive

$$\triangleright \text{i.} \quad t \simeq t \quad (\text{eq_reflexive})$$

25. eq_transitive

$$\triangleright \text{i.} \quad \neg(t_1 \simeq t_2) \dot{\vee} \dots \dot{\vee} \neg(t_{n-1} \simeq t_n) \dot{\vee} t_1 \simeq t_n \quad (\text{eq_transitive})$$

26. eq_congruent

$$\triangleright \text{i.} \quad \neg(t_1 \simeq u_1) \dot{\vee} \dots \dot{\vee} \neg(t_n \simeq u_n) \dot{\vee} f(t_1, \dots, t_n) \simeq f(u_1, \dots, u_n) \quad (\text{eq_congruent})$$

27. eq_congruent_pred

$$\triangleright \text{i.} \quad \neg(t_1 \simeq u_1) \dot{\vee} \dots \dot{\vee} \neg(t_n \simeq u_n) \dot{\vee} P(t_1, \dots, t_n) \simeq P(u_1, \dots, u_n) \quad (\text{eq_congruent_pred})$$

28. distinct_elim

This rule eliminates the distinct predicate. If called with one argument this predicate always holds:

$$\triangleright \text{i.} \quad (\text{distinct } t) \leftrightarrow \top \quad (\text{distinct_elim})$$

If applied to terms of type **Bool** more than two terms can never be distinct, hence only two cases are possible:

$$\triangleright \text{i.} \quad (\text{distinct } \varphi \ \psi) \leftrightarrow \neg(\varphi \leftrightarrow \psi) \quad (\text{distinct_elim})$$

and

$$\triangleright \text{i.} \quad (\text{distinct } \varphi_1 \ \varphi_2 \ \varphi_3 \dots) \leftrightarrow \perp \quad (\text{distinct_elim})$$

The general case is:

$$\triangleright \text{i.} \quad (\text{distinct } t_1 \dots t_n) \leftrightarrow \bigwedge_{i=1}^n \bigwedge_{j=i+1}^n t_i \not\approx t_j \quad (\text{distinct_elim})$$

29. la_rw_eq

$$\triangleright \text{i.} \quad (t \simeq u) \simeq (t \leq u \wedge u \leq t) \quad (\text{la_rw_eq})$$

Remark. While the connective could be an \leftrightarrow , currently an equality is used.

30. la_generic

A step of the `la_generic` rule represents a tautological clause of linear disequalities. It can be checked by showing that the conjunction of the negated disequalities is unsatisfiable. After

the application of some strengthening rules, the resulting conjunction is unsatisfiable, even if integer variables are assumed to be real variables.

A linear inequality is of term of the form $\sum_{i=0}^n c_i \times t_i + d_1 \bowtie \sum_{i=n+1}^m c_i \times t_i + d_2$ where $\bowtie \in \{=, <, >, \leq, \geq\}$, where $m \geq n$, c_i, d_1, d_2 are either integer or real constants, and for each i c_i and t_i have the same sort. We will write $s_1 \bowtie s_2$.

Let l_1, \dots, l_n be linear inequalities and a_1, \dots, a_n rational numbers, then a `la_generic` step has the form:

$$\triangleright i. \ \varphi_1 \dot{\vee} \dots \dot{\vee} \varphi_o \quad (\text{la_generic};; a_1, \dots, a_o)$$

where φ_i is either $\neg l_i$ or l_i , but never $s_1 \simeq s_2$.

If the current theory does not have rational numbers, then the a_i are printed using integer division. They should, nevertheless, be interpreted as rational numbers. If d_1 or d_2 are 0, they might not be printed.

To check the unsatisfiability of the negation of $\varphi_1 \vee \dots \vee \varphi_o$ one performs the following steps for each literal. For each i , let $\varphi := \varphi_i$ and $a := a_i$.

1. If $\varphi = s_1 > s_2$, then let $\varphi := s_1 \leq s_2$. If $\varphi = s_1 \geq s_2$, then let $\varphi := s_1 < s_2$. If $\varphi = s_1 < s_2$, then let $\varphi := s_1 \geq s_2$. If $\varphi = s_1 \leq s_2$, then let $\varphi := s_1 > s_2$.
2. If $\varphi = \neg(s_1 \bowtie s_2)$, then let $\varphi := s_1 \bowtie s_2$.
3. Replace φ by $\sum_{i=0}^n c_i \times t_i - \sum_{i=n+1}^m c_i \times t_i \bowtie d$ where $d := d_2 - d_1$.
4. Now φ has the form $s_1 \bowtie d$. If all variables in s_1 are integer sorted: replace $\bowtie d$ according to table 1.
5. If \bowtie is \simeq replace l by $\sum_{i=0}^m a \times c_i \times t_i \simeq a \times d$, otherwise replace it by $\sum_{i=0}^m |a| \times c_i \times t_i \simeq |a| \times d$.

\bowtie	If d is an integer	Otherwise
$>$	$\geq d + 1$	$\geq \lfloor d \rfloor + 1$
\geq	$\geq d$	$\geq \lfloor d \rfloor + 1$

Table 1: Strengthening rules for `la_generic`.

Finally, the sum of the resulting literals is trivially contradictory. The sum

$$\sum_{k=1}^o \sum_{i=1}^{m^o} c_i^o * t_i^o \bowtie \sum_{k=1}^o d^k$$

where c_i^k is the constant c_i of literal l_k , t_i^k is the term t_i of l_k , and d^k is the constant d of l_k . The operator \bowtie is \simeq if all operators are \simeq , $>$ if all are either \simeq or $>$, and \geq otherwise. The a_i must be such that the sum on the left-hand side is 0 and the right-hand side is > 0 (or ≥ 0 if \bowtie is $>$).

Example 30.1. A simple `la_generic` step in the logic LRA might look like this:

```
(step t10 (cl (not (> (f a) (f b))) (not (= (f a) (f b))))
  :rule la_generic :args (1.0 (- 1.0)))
```

To verify this we have to check the insatisfiability of $f(a) > f(b) \wedge f(a) = f(b)$ (Step 2). After step 3 we get $f(a) - f(b) > 0 \wedge f(a) - f(b) = 0$. Since we don't have an integer sort in this logic step 4 does not apply. Finally, after step 5 the conjunction is $f(a) - f(b) > 0 \wedge -f(a) + f(b) = 0$. This sums to $0 > 0$, which is a contradiction.

Example 30.2. The following `la_generic` step is from a QF_UFLIA problem:

```
(step t11 (cl (not (<= f3 0)) (<= (+ 1 (* 4 f3)) 1))
  :rule la_generic :args (1 (div 1 4)))
```

After normalization we get $-f_3 \geq 0 \wedge 4 \times f_3 > 0$. This time step 4 applies and we can strengthen this to $-f_3 \geq 0 \wedge 4 \times f_3 \geq 1$ and after multiplication we get $-f_3 \geq 0 \wedge f_3 \geq \frac{1}{4}$. Which sums to the contradiction $\frac{1}{4} \geq 0$.

31. `lia_generic`

This rule is a placeholder rule for integer arithmetic solving. It takes the same form as `la_generic`, without the additional arguments.

$$\triangleright i. \quad \varphi_1 \dot{\vee} \dots \dot{\vee} \varphi_n \quad (\text{lia_generic})$$

with φ_i being linear inequalities. The disjunction $\varphi_1 \dot{\vee} \dots \dot{\vee} \varphi_n$ is a tautology in the theory of linear integer arithmetic.

32. `la_disequality`

$$\triangleright i. \quad t_1 \simeq t_2 \vee \neg(t_1 \leq t_2) \vee \neg(t_2 \leq t_1) \quad (\text{la_disequality})$$

33. `la_totality`

$$\triangleright i. \quad t_1 \leq t_2 \vee t_2 \leq t_1 \quad (\text{la_totality})$$

34. `la_tautology`

This rule is a linear arithmetic tautology which can be checked without sophisticated reasoning. It has either the form:

$$\triangleright i. \quad \varphi \quad (\text{la_tautology})$$

where φ is either a linear inequality $s_1 \bowtie s_2$ or $\neg(s_1 \bowtie s_2)$. After performing step 1 to 3 of the process for checking the `la_generic` the result is trivially unsatisfiable.

The second form handles bounds on linear combinations. It is binary clause:

▷ i. $\varphi_1 \vee \varphi_2$ (la_tautology)

It can be checked by using the procedure for `la_generic` with while setting the arguments to 1. Informally, the rule follows one of several cases:

- $\neg(s_1 \leq d_1) \vee s_1 \leq d_2$ where $d_1 \leq d_2$
- $s_1 \leq d_1 \vee \neg(s_1 \leq d_2)$ where $d_1 = d_2$
- $\neg(s_1 \geq d_1) \vee s_1 \geq d_2$ where $d_1 \geq d_2$
- $s_1 \geq d_1 \vee \neg(s_1 \geq d_2)$ where $d_1 = d_2$
- $\neg(s_1 \leq d_1) \vee \neg(s_1 \geq d_2)$ where $d_1 < d_2$

The inequalities $s_1 \bowtie d$ are the result of applying normalization as for the rule `la_generic`.

35. forall_inst

▷ i. $\neg(\forall x_1, \dots, x_n. P) \vee P[t_1/x_1] \dots [t_n/x_n]$ (forall_inst; $x_{k_1} := t_{k_1}, \dots, x_{k_n} := t_{k_n}$)

where k_1, \dots, k_n is a permutation of $1, \dots, n$ and x_i and k_i have the same sort. The arguments $x_{k_i} := t_{k_i}$ are printed as `(:= xki tki)`.

Remark. A rule similar to the `let` rule would be more appropriate. The resulting proof would be more fine grained and this would also be an opportunity to provide a proof for the classification as currently done by `qnt_cnf`.

36. qnt_join

Γ ▷ i. $Qx_1, \dots, x_n. Qx_{n+1}, \dots, x_m. \varphi \leftrightarrow Qx_{k_1}, \dots, x_{k_o}. \varphi$ (qnt_join)

where $m > n$, $Q \in \{\forall, \exists\}$, k_1, \dots, k_o is monotonic map to $1, \dots, m$ such that x_{k_1}, \dots, x_{k_o} are pairwise distinct, and $\{x_1, \dots, x_m\} = \{x_{k_1}, \dots, x_{k_o}\}$.

37. qnt_rm_unused

Γ ▷ i. $Qx_1, \dots, x_n. \varphi \leftrightarrow Qx_{k_1}, \dots, x_{k_m}. \varphi$ (qnt_rm_unused)

where $m \leq n$, $Q \in \{\forall, \exists\}$, k_1, \dots, k_m is monotonic map to $1, \dots, n$ and if $x \in \{x_j \mid j \in \{1, \dots, n\} \wedge j \notin \{k_1, \dots, k_m\}\}$ then x is not free in P .

38. th_resolution

This rule is the resolution of two or more clauses.

$$\begin{array}{lll}
\triangleright i^1. & \varphi_1^1 \dot{\vee} \dots \dot{\vee} \varphi_{k^1}^1 & (\dots) \\
& \vdots & \\
\triangleright i^n. & \varphi_1^n \dot{\vee} \dots \dot{\vee} \varphi_{k^n}^n & (\dots) \\
& \vdots & \\
\triangleright j. & \varphi_{s_1}^{r_1} \dot{\vee} \dots \dot{\vee} \varphi_{s_m}^{r_m} & (\text{th_resolution}; i^1, \dots, i^n)
\end{array}$$

where $\varphi_{s_1}^{r_1} \dot{\vee} \dots \dot{\vee} \varphi_{s_m}^{r_m}$ are from φ_j^i and are the result of a chain of predicate resolution steps on the clauses i^1 to i^n . It is possible that $m = 0$, i.e. that the result is the empty clause.

This rule is only used when the resolution step is not emitted by the SAT solver. See the equivalent resolution rule for the rule emitted by the SAT solver.

Remark. While checking this rule is NP-complete, the `th_resolution`-steps produced by veriT are simple. Experience with reconstructing the step in Isabelle/HOL shows that checking can be done by naive decision procedures. The vast majority of `th_resolution`-steps are binary resolution steps.

39. resolution

This rule is equivalent to the `the_resolution` rule, but it is emitted by the SAT solver instead of theory reasoners. The differentiation serves only informational purpose.

40. refl

Either

$$\Gamma \triangleright j. \quad t_1 \simeq t_2 \quad (\text{refl})$$

or

$$\Gamma \triangleright j. \quad \varphi_1 \leftrightarrow \varphi_2 \quad (\text{refl})$$

where φ_1 and φ_2 (P_1 and P_2) are equal after applying the substitution induced by Γ .

41. trans

Either

$$\Gamma \triangleright i. \quad t_1 \simeq t_2 \quad (\dots)$$

\vdots

$$\Gamma \triangleright j. \quad t_2 \simeq t_3 \quad (\dots)$$

\vdots

$$\Gamma \triangleright k. \quad t_1 \simeq t_3 \quad (\text{trans}; i, j)$$

or

$$\begin{array}{lll}
\Gamma \triangleright i. & \varphi_1 \leftrightarrow \varphi_2 & (\dots) \\
& \vdots & \\
\Gamma \triangleright j. & \varphi_2 \leftrightarrow \varphi_3 & (\dots) \\
& \vdots & \\
\Gamma \triangleright k. & \varphi_1 \leftrightarrow \varphi_3 & (\text{trans}; i, j)
\end{array}$$

42. cong

Either

$$\begin{array}{lll}
\Gamma \triangleright i_1. & t_1 \simeq u_1 & (\dots) \\
& \vdots & \\
\Gamma \triangleright i_n. & t_n \simeq u_n & (\dots) \\
& \vdots & \\
\Gamma \triangleright j. & f(t_1, \dots, t_n) \simeq f(u_1, \dots, u_n) & (\text{cong}; i_1, \dots, i_n)
\end{array}$$

where f is an n -ary function symbol, or

$$\begin{array}{lll}
\Gamma \triangleright i_1. & \varphi_1 \simeq \psi_1 & (\dots) \\
& \vdots & \\
\Gamma \triangleright i_n. & \varphi_n \simeq \psi_n & (\dots) \\
& \vdots & \\
\Gamma \triangleright j. & P(\varphi_1, \dots, \varphi_n) \leftrightarrow P(\psi_1, \dots, \psi_n) & (\text{cong}; i_1, \dots, i_n)
\end{array}$$

where P is an n -ary predicate symbol.

43. and

$$\begin{array}{lll}
\triangleright i. & \varphi_1 \wedge \dots \wedge \varphi_n & (\dots) \\
& \vdots & \\
\triangleright j. & \varphi_i & (\text{and}; i)
\end{array}$$

44. tautology

$$\begin{array}{lll}
\triangleright i. & \varphi_1 \dot{\vee} \dots \dot{\vee} \varphi_i \dot{\vee} \dots \dot{\vee} \varphi_j \dot{\vee} \dots \dot{\vee} \varphi_n & (\dots) \\
& \vdots & \\
\triangleright j. & \top & (\text{tautology}; i)
\end{array}$$

and $\varphi_i = \bar{\varphi}_j$.

45. not_or

$$\begin{array}{ll}
\triangleright \text{i.} & \neg(\varphi_1 \vee \dots \vee \varphi_n) \quad (\dots) \\
& \vdots \\
\triangleright \text{j.} & \neg\varphi_i \quad (\text{not_or}; i)
\end{array}$$

46. or

$$\begin{array}{ll}
\triangleright \text{i.} & \varphi_1 \vee \dots \vee \varphi_n \quad (\dots) \\
& \vdots \\
\triangleright \text{j.} & \varphi_1 \dot{\vee} \dots \dot{\vee} \varphi_n \quad (\text{or}; i)
\end{array}$$

Remark. This rule deconstructs the `or` operator into a `cl`.

Example 46.1. An application of the `or` rule.

```

(step t15 (cl (or (= a b) (not (<= a b)) (not (<= b a))))
  :rule la_disequality)
(step t16 (cl (= a b) (not (<= a b)) (not (<= b a)))
  :rule or :premises (t15))

```

47. not_and

$$\begin{array}{ll}
\triangleright \text{i.} & \neg(\varphi_1 \wedge \dots \wedge \varphi_n) \quad (\dots) \\
& \vdots \\
\triangleright \text{j.} & \neg\varphi_1 \dot{\vee} \dots \dot{\vee} \neg\varphi_n \quad (\text{not_and}; i)
\end{array}$$

48. xor1

$$\begin{array}{ll}
\triangleright \text{i.} & \text{xor } \varphi_1 \varphi_2 \quad (\dots) \\
& \vdots \\
\triangleright \text{j.} & \varphi_1 \dot{\vee} \varphi_2 \quad (\text{xor1}; i)
\end{array}$$

49. xor2

$$\begin{array}{ll}
\triangleright \text{i.} & \text{xor } \varphi_1 \varphi_2 \quad (\dots) \\
& \vdots \\
\triangleright \text{j.} & \neg\varphi_1 \dot{\vee} \neg\varphi_2 \quad (\text{xor2}; i)
\end{array}$$

50. not_xor1

▷ i. $\neg(\text{xor } \varphi_1 \varphi_2)$ (...)

▷ j. $\varphi_1 \dot{\vee} \neg\varphi_2$ (not_xor1;i)

51. not_xor2

▷ i. $\neg(\text{xor } \varphi_1 \varphi_2)$ (...)

▷ j. $\neg\varphi_1 \dot{\vee} \varphi_2$ (not_xor2;i)

52. implies

▷ i. $\varphi_1 \rightarrow \varphi_2$ (...)

▷ j. $\neg\varphi_1 \dot{\vee} \varphi_2$ (implies;i)

53. not_implies1

▷ i. $\neg(\varphi_1 \rightarrow \varphi_2)$ (...)

▷ j. φ_1 (not_implies1;i)

54. not_implies2

▷ i. $\neg(\varphi_1 \rightarrow \varphi_2)$ (...)

▷ j. $\neg\varphi_2$ (not_implies2;i)

55. equiv1

▷ i. $\varphi_1 \leftrightarrow \varphi_2$ (...)

▷ j. $\neg\varphi_1 \dot{\vee} \varphi_2$ (equiv1;i)

56. equiv2

▷ i. $\varphi_1 \leftrightarrow \varphi_2$ (...)

▷ j. $\varphi_1 \dot{\vee} \neg\varphi_2$ (equiv2;i)

57. not_equiv1

$$\begin{array}{lll}
\triangleright i. & \neg(\varphi_1 \leftrightarrow \varphi_2) & (\dots) \\
& \vdots & \\
\triangleright j. & \varphi_1 \dot{\vee} \varphi_2 & (\text{not_equiv1}; i)
\end{array}$$

58. not_equiv2

$$\begin{array}{lll}
\triangleright i. & \neg(\varphi_1 \leftrightarrow \varphi_2) & (\dots) \\
& \vdots & \\
\triangleright j. & \neg\varphi_1 \dot{\vee} \neg\varphi_2 & (\text{not_equiv2}; i)
\end{array}$$

59. ite1

$$\begin{array}{lll}
\triangleright i. & \text{ite } \varphi_1 \varphi_2 \varphi_3 & (\dots) \\
& \vdots & \\
\triangleright j. & \varphi_1 \dot{\vee} \varphi_3 & (\text{ite1}; i)
\end{array}$$

60. ite2

$$\begin{array}{lll}
\triangleright i. & \text{ite } \varphi_1 \varphi_2 \varphi_3 & (\dots) \\
& \vdots & \\
\triangleright j. & \neg\varphi_1 \dot{\vee} \varphi_2 & (\text{ite2}; i)
\end{array}$$

61. not_ite1

$$\begin{array}{lll}
\triangleright i. & \neg(\text{ite } \varphi_1 \varphi_2 \varphi_3) & (\dots) \\
& \vdots & \\
\triangleright j. & \varphi_1 \dot{\vee} \neg\varphi_3 & (\text{not_ite1}; i)
\end{array}$$

62. not_ite2

$$\begin{array}{lll}
\triangleright i. & \neg(\text{ite } \varphi_1 \varphi_2 \varphi_3) & (\dots) \\
& \vdots & \\
\triangleright j. & \neg\varphi_2 \dot{\vee} \neg\varphi_2 & (\text{not_ite2}; i)
\end{array}$$

63. ite_intro

Either

$$\triangleright i. \quad t \simeq (t' \wedge u_1 \wedge \dots \wedge u_n) \quad (\dots)$$

or

$$\triangleright \text{i.} \quad \varphi \leftrightarrow (\varphi' \wedge u_1 \wedge \dots \wedge u_n) \quad (\dots)$$

The term t (the formula φ) contains the ite operator. Let s_1, \dots, s_n be the terms starting with ite, i.e. $s_i := \text{ite } \psi_i \ r_i \ r'_i$, then u_i has the form:

$$\text{ite } \psi_i \ (s_i \simeq r_i) \ (s_i \simeq r'_i)$$

or

$$\text{ite } \psi_i \ (s_i \leftrightarrow r_i) \ (s_i \leftrightarrow r'_i)$$

if s_i is of sort **Bool**. The term t' (the formula φ') is equal to the term t (the formula φ) up to the reordering of equalities where one argument is an ite term.

Remark. This rule stems from the introduction of fresh constants for if-then-else terms inside veriT. Internally s_i is a new constant symbol and the φ on the right side of the equality is φ with the if-then-else terms replaced by the constants. Those constants are unfolded during proof printing. Hence, the slightly strange form and the reordering of equalities.

64. contraction

$$\triangleright \text{i.} \quad \varphi_1 \dot{\vee} \dots \dot{\vee} \varphi_n \quad (\dots)$$

$$\triangleright \text{j.} \quad \begin{array}{c} \vdots \\ \varphi_{k_1} \dot{\vee} \dots \dot{\vee} \varphi_{k_m} \end{array} \quad (\text{contraction}; \text{i})$$

where $m \leq n$ and $k_1 \dots k_m$ is a monotonic map to $1 \dots n$ such that $\varphi_{k_1} \dots \varphi_{k_m}$ are pairwise distinct and $\{\varphi_1, \dots, \varphi_n\} = \{\varphi_{k_1} \dots \varphi_{k_m}\}$. Hence, this rule remove duplicated literals.

65. connective_def

This rule is used to replace connectives by their definition. It can be one of the following:

$$\Gamma \triangleright \text{i.} \quad \varphi_1 \text{ xor } \varphi_2 \leftrightarrow (\neg \varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \neg \varphi_2) \quad (\text{connective_def})$$

$$\Gamma \triangleright \text{i.} \quad \varphi_1 \leftrightarrow \varphi_2 \leftrightarrow (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1) \quad (\text{connective_def})$$

$$\Gamma \triangleright \text{i.} \quad \text{ite } \varphi_1 \ \varphi_2 \ \varphi_3 \leftrightarrow (\varphi_1 \rightarrow \varphi_2) \wedge (\neg \varphi_1 \rightarrow \neg \varphi_3) \quad (\text{connective_def})$$

66. ite_simplify

This rule simplifies an if-then-else term by applying equivalent transformations as long as possible. Depending on the sort of the ite-term the rule can have one of two forms. If the sort is **Bool** it has the form

$$\Gamma \triangleright \text{i.} \quad \text{ite } \varphi \ t_1; t_2 \leftrightarrow \psi \quad (\text{ite_simplify})$$

with ψ being the transformed term.

Otherwise, it has the form

$\Gamma \triangleright i.$ $\text{ite } \varphi \ t_1 \ t_2 \simeq u$ (ite_simplify)

with u being the transformed term.

The possible transformations are:

- $\text{ite } \top \ t_1 \ t_2 \leftrightarrow t_1$
- $\text{ite } \perp \ t_1 \ t_2 \leftrightarrow t_2$
- $\text{ite } \psi \ t \ t \leftrightarrow t$
- $\text{ite } \neg\varphi \ t_1 \ t_2 \leftrightarrow \text{ite } \varphi \ t_2 \ t_1$
- $\text{ite } \top \ t_1 \ t_2 \leftrightarrow t_1$
- $\text{ite } \perp \ t_1 \ t_2 \leftrightarrow t_2$
- $\text{ite } \psi \ (\text{ite } \psi \ t_1 \ t_2) \ t_3 \leftrightarrow \text{ite } \psi \ t_1 \ t_3$
- $\text{ite } \psi \ t_1 \ (\text{ite } \psi \ t_2 \ t_3) \leftrightarrow \text{ite } \psi \ t_1 \ t_3$
- $\text{ite } \psi \ \top \ \perp \leftrightarrow \psi$
- $\text{ite } \psi \ \perp \ \top \leftrightarrow \neg\psi$
- $\text{ite } \psi \ \top \ \varphi \leftrightarrow \psi \vee \varphi$
- $\text{ite } \psi \ \varphi \ \perp \leftrightarrow \psi \wedge \varphi$
- $\text{ite } \psi \ \perp \ \varphi \leftrightarrow \neg\psi \wedge \varphi$
- $\text{ite } \psi \ \varphi \ \top \leftrightarrow \neg\psi \vee \varphi$

67. eq_simplify

This rule simplifies an \simeq term by applying equivalent transformations as long as possible. Hence, the general form is:

$\Gamma \triangleright i.$ $t_1 \simeq t_2 \leftrightarrow \varphi$ (eq_simplify)

with ψ being the transformed term.

The possible transformations are:

- $t \simeq t \leftrightarrow \top$
- $t_1 \simeq t_2 \leftrightarrow \perp$ if t_1 and t_2 are different numeric constants.
- $\neg(t \simeq t) \leftrightarrow \perp$ if t is a numeric constant.

68. and_simplify

This rule simplifies an \wedge term by applying equivalent transformations as long as possible. Hence, the general form is:

$$\Gamma \triangleright i. \quad \varphi_1 \wedge \cdots \wedge \varphi_n \leftrightarrow \psi \quad (\text{and_simplify})$$

with ψ being the transformed term.

The possible transformations are:

- $\top \wedge \cdots \wedge \top \leftrightarrow \top$
- $\varphi_1 \wedge \cdots \wedge \varphi_n \leftrightarrow \varphi_1 \wedge \cdots \wedge \varphi_{n'}$ where the right hand side has all \top literals removed.
- $\varphi_1 \wedge \cdots \wedge \varphi_n \leftrightarrow \varphi_1 \wedge \cdots \wedge \varphi_{n'}$ where the right hand side has all repeated literals removed.
- $\varphi_1 \wedge \cdots \wedge \perp \wedge \cdots \wedge \varphi_n \leftrightarrow \perp$
- $\varphi_1 \wedge \cdots \wedge \varphi_i \wedge \cdots \wedge \varphi_j \wedge \cdots \wedge \varphi_n \leftrightarrow \perp$ if $\varphi_i = \bar{\varphi}_j$

69. or_simplify

This rule simplifies an \vee term by applying equivalent transformations as long as possible. Hence, the general form is:

$$\Gamma \triangleright i. \quad (\varphi_1 \vee \cdots \vee \varphi_n) \leftrightarrow \psi \quad (\text{or_simplify})$$

with ψ being the transformed term.

The possible transformations are:

- $\perp \vee \cdots \vee \perp \leftrightarrow \perp$
- $\varphi_1 \vee \cdots \vee \varphi_n \leftrightarrow \varphi_1 \vee \cdots \vee \varphi_{n'}$ where the right hand side has all \perp literals removed.
- $\varphi_1 \vee \cdots \vee \varphi_n \leftrightarrow \varphi_1 \vee \cdots \vee \varphi_{n'}$ where the right hand side has all repeated literals removed.
- $\varphi_1 \vee \cdots \vee \top \vee \cdots \vee \varphi_n \leftrightarrow \top$
- $\varphi_1 \vee \cdots \vee \varphi_i \vee \cdots \vee \varphi_j \vee \cdots \vee \varphi_n \leftrightarrow \top$ if $\varphi_i = \bar{\varphi}_j$

70. not_simplify

This rule simplifies an \neg term by applying equivalent transformations as long as possible. Hence, the general form is:

$$\Gamma \triangleright i. \quad \neg \varphi \leftrightarrow \psi \quad (\text{not_simplify})$$

with ψ being the transformed term.

The possible transformations are:

- $\neg(\neg\varphi) \leftrightarrow \varphi$
- $\neg\perp \leftrightarrow \top$
- $\neg\top \leftrightarrow \perp$

71. implies_simplify

This rule simplifies an \rightarrow term by applying equivalent transformations as long as possible. Hence, the general form is:

$$\Gamma \triangleright i. \quad \varphi_1 \rightarrow \varphi_2 \leftrightarrow \psi \quad (\text{implies_simplify})$$

with ψ being the transformed term.

The possible transformations are:

- $\neg\varphi_1 \rightarrow \neg\varphi_2 \leftrightarrow \varphi_2 \rightarrow \varphi_1$
- $\perp \rightarrow \varphi \leftrightarrow \top$
- $\varphi \rightarrow \top \leftrightarrow \top$
- $\top \rightarrow \varphi \leftrightarrow \varphi$
- $\varphi \rightarrow \perp \leftrightarrow \neg\varphi$
- $\varphi \rightarrow \varphi \leftrightarrow \top$
- $\neg\varphi \rightarrow \varphi \leftrightarrow \varphi$
- $\varphi \rightarrow \neg\varphi \leftrightarrow \neg\varphi$
- $(\varphi_1 \rightarrow \varphi_2) \rightarrow \varphi_2 \leftrightarrow \varphi_1 \vee \varphi_2$

72. equiv_simplify

This rule simplifies an \leftrightarrow term by applying equivalent transformations as long as possible. Hence, the general form is:

$$\Gamma \triangleright i. \quad \varphi_1 \leftrightarrow \varphi_2 \leftrightarrow \psi \quad (\text{equiv_simplify})$$

with ψ being the transformed term.

The possible transformations are:

- $(\neg\varphi_1 \leftrightarrow \neg\varphi_2) \leftrightarrow (\varphi_1 \leftrightarrow \varphi_2)$
- $(\varphi \leftrightarrow \varphi) \leftrightarrow \top$

- $(\varphi \leftrightarrow \neg\varphi) \leftrightarrow \perp$
- $(\neg\varphi \leftrightarrow \varphi) \leftrightarrow \perp$
- $(\top \leftrightarrow \varphi) \leftrightarrow \varphi$
- $(\varphi \leftrightarrow \top) \leftrightarrow \varphi$
- $(\perp \leftrightarrow \varphi) \leftrightarrow \neg\varphi$
- $(\varphi \leftrightarrow \perp) \leftrightarrow \neg\varphi$

73. bool_simplify

This rule simplifies a boolean term by applying equivalent transformations as long as possible. Hence, the general form is:

$$\Gamma \triangleright i. \quad \varphi \leftrightarrow \psi \quad (\text{bool_simplify})$$

with ψ being the transformed term.

The possible transformations are:

- $\neg(\varphi_1 \rightarrow \varphi_2) \leftrightarrow (\varphi_1 \wedge \neg\varphi_2)$
- $\neg(\varphi_1 \vee \varphi_2) \leftrightarrow (\neg\varphi_1 \wedge \neg\varphi_2)$
- $\neg(\varphi_1 \wedge \varphi_2) \leftrightarrow (\neg\varphi_1 \vee \neg\varphi_2)$
- $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)) \leftrightarrow (\varphi_1 \wedge \varphi_2) \rightarrow \varphi_3$
- $((\varphi_1 \rightarrow \varphi_2) \rightarrow \varphi_2) \leftrightarrow (\varphi_1 \vee \varphi_2)$
- $(\varphi_1 \wedge (\varphi_1 \rightarrow \varphi_2)) \leftrightarrow (\varphi_1 \wedge \varphi_2)$
- $((\varphi_1 \rightarrow \varphi_2) \wedge \varphi_1) \leftrightarrow (\varphi_1 \wedge \varphi_2)$

74. qnt_simplify

This rule simplifies a \forall term with a constant predicate.

$$\Gamma \triangleright i. \quad \forall x_1, \dots, x_n. \varphi \leftrightarrow \varphi \quad (\text{qnt_simplify})$$

where φ is either \top or \perp .

75. div_simplify

This rule simplifies a division by applying equivalent transformations. The general form is:

$$\Gamma \triangleright i. \quad \frac{t_1}{t_2} \simeq t_3 \quad (\text{div_simplify})$$

The possible transformations are:

- $\frac{t}{t} = 1$
- $\frac{t}{1} = t$
- $\frac{t_1}{t_2} = t_3$ if t_1 and t_2 are constants and t_3 is t_1 divided by t_2 according to the semantic of the current theory.

76. prod_simplify

This rule simplifies a product by applying equivalent transformations as long as possible. The general form is:

$$\Gamma \triangleright i. \quad t_1 \times \cdots \times t_n \simeq u \quad (\text{prod_simplify})$$

where u is either a constant or a product.

The possible transformations are:

- $t_1 \times \cdots \times t_n = u$ where all t_i are constants and u is their product.
- $t_1 \times \cdots \times t_n = 0$ if any t_i is 0.
- $t_1 \times \cdots \times t_n = c \times t_{k_1} \times \cdots \times t_{k_n}$ where c is the product of the constants of t_1, \dots, t_n and t_{k_1}, \dots, t_{k_n} is t_1, \dots, t_n with the constants removed.
- $t_1 \times \cdots \times t_n = t_{k_1} \times \cdots \times t_{k_n}$: same as above if c is 1.

77. unary_minus_simplify

This rule is either

$$\Gamma \triangleright i. \quad -(-t) \simeq t \quad (\text{unary_minus_simplify})$$

or

$$\Gamma \triangleright i. \quad -t \simeq u \quad (\text{unary_minus_simplify})$$

where u is the negated numerical constant t .

78. minus_simplify

This rule simplifies a subtraction by applying equivalent transformations. The general form is:

$$\Gamma \triangleright i. \quad t_1 - t_2 \simeq u \quad (\text{minus_simplify})$$

The possible transformations are:

- $t - t = 0$

- $t_1 - t_2 = t_3$ where t_1 and t_2 are numerical constants and t_3 is t_2 subtracted from t_1 .
- $t - 0 = t$
- $0 - t = -t$

79. sum_simplify

This rule simplifies a sum by applying equivalent transformations as long as possible. The general form is:

$$\Gamma \triangleright i. \quad t_1 + \cdots + t_n \simeq u \quad (\text{sum_simplify})$$

where u is either a constant or a product.

The possible transformations are:

- $t_1 + \cdots + t_n = c$ where all t_i are constants and c is their sum.
- $t_1 + \cdots + t_n = c + t_{k_1} + \cdots + t_{k_n}$ where c is the sum of the constants of t_1, \dots, t_n and t_{k_1}, \dots, t_{k_n} is t_1, \dots, t_n with the constants removed.
- $t_1 + \cdots + t_n = t_{k_1} + \cdots + t_{k_n}$: same as above if c is 0.

80. comp_simplify

This rule simplifies a comparison by applying equivalent transformations as long as possible. The general form is:

$$\Gamma \triangleright i. \quad t_1 \bowtie t_n \leftrightarrow \psi \quad (\text{comp_simplify})$$

where \bowtie is as for the proof rule `la_generic`, but never \simeq .

The possible transformations are:

- $t_1 < t_2 \leftrightarrow \varphi$ where t_1 and t_2 are numerical constants and φ is \top if t_1 is strictly greater than t_2 and \perp otherwise.
- $t < t \leftrightarrow \perp$
- $t_1 \leq t_2 \leftrightarrow \varphi$ where t_1 and t_2 are numerical constants and φ is \top if t_1 is greater than t_2 or equal and \perp otherwise.
- $t \leq t \leftrightarrow \top$
- $t_1 \geq t_2 \leftrightarrow t_2 \leq t_1$
- $t_1 < t_2 \leftrightarrow \neg(t_2 \leq t_1)$
- $t_1 > t_2 \leftrightarrow \neg(t_1 \leq t_2)$

81. nary_elim

This rule replaces n -ary operators with their equivalent application of the binary operator. It is never applied to \wedge or \vee .

Three cases are possible. If the operator \circ is left associative, then the rule has the form

$$\Gamma \triangleright i. \quad \bigcirc_{i=1}^n t_i \leftrightarrow (\dots (t_1 \circ t_2) \circ t_3) \circ \dots t_n \quad (\text{nary_elim})$$

If the operator \circ is right associative, then the rule has the form

$$\Gamma \triangleright i. \quad \bigcirc_{i=1}^n t_i \leftrightarrow (t_1 \circ \dots \circ (t_{n-2} \circ (t_{n-1} \circ t_n) \dots)) \quad (\text{nary_elim})$$

If the operator is *chainable*, then it has the form

$$\Gamma \triangleright i. \quad \bigcirc_{i=1}^n t_i \leftrightarrow (t_1 \circ t_2) \wedge (t_2 \circ t_3) \wedge \dots \wedge (t_{n-1} \circ t_n) \quad (\text{nary_elim})$$

82. ac_simp

This rule simplifies nested occurrences of \vee or \wedge :

$$\Gamma \triangleright i. \quad \psi \leftrightarrow \varphi_1 \circ \dots \circ \varphi_n \quad (\text{ac_simp})$$

where $\circ \in \{\vee, \wedge\}$ and ψ is a nested application of \circ . The literals φ_i are literals of the flattening of ψ with duplicates removed.

83. bfun_elim

$$\begin{array}{l} \triangleright i. \quad \psi \quad (\dots) \\ \quad \quad \quad \vdots \\ \triangleright j. \quad \varphi \quad (\text{bfun_elim}; i) \end{array}$$

The formula φ is ψ after boolean functions have been simplified. This happens in a two step process. Both steps recursively iterate over ψ . The first step expands quantified variable of type **Bool**. Hence, $\exists x.t$ becomes $t[\perp/x] \vee t[\top/x]$ and $\forall x.t$ becomes $t[\perp/x] \wedge t[\top/x]$. If n variables of sort **Bool** appear in a quantifier, the disjunction (conjunction) has 2^n terms. Each term replaces the variables in t according to the bits of a number which is increased by one for each subsequent term starting from zero. The left-most variable corresponds to the least significant bit.

The second step expands function argument of boolean types by introducing appropriate if-then-else terms. For example, consider $f(x, P, y)$ where P is some formula. Then we replace this term by $\text{ite } P \ f(x, \top, y) \ f(x, \perp, y)$. If the argument is already the constant \top or \perp it is ignored.

84. deep_skolemize

This rule is only emitted when the option `--enable-deep-skolem` is given. This option is experimental and should not be used.

85. qnt_cnf

$$\triangleright \text{i.} \quad \neg(\forall x_1, \dots, x_n. \varphi) \vee \forall x_{k_1}, \dots, x_{k_m}. \varphi' \quad (\text{qnt_cnf})$$

This is a placeholder rule for clausification of a term under a universal quantifier. This is used by conflicting instantiation. φ' is one of the clause of the clause normal form of φ . The variables x_{k_1}, \dots, x_{k_m} are a permutation of x_1, \dots, x_n plus additional variables added by prefixing φ . Normalization is performed in two phases. First, the negative normal form is formed, then the result is prefixed. The result of the first step is $\Phi(\varphi, 1)$ where:

$$\begin{aligned} \Phi(\neg\varphi, 1) &:= \Phi(\varphi, 0) \\ \Phi(\neg\varphi, 0) &:= \Phi(\varphi, 1) \\ \Phi(\varphi_1 \vee \dots \vee \varphi_n, 1) &:= \Phi(\varphi_1, 1) \vee \dots \vee \Phi(\varphi_n, 1) \\ \Phi(\varphi_1 \wedge \dots \wedge \varphi_n, 1) &:= \Phi(\varphi_1, 1) \wedge \dots \wedge \Phi(\varphi_n, 1) \\ \Phi(\varphi_1 \vee \dots \vee \varphi_n, 0) &:= \Phi(\varphi_1, 0) \wedge \dots \wedge \Phi(\varphi_n, 0) \\ \Phi(\varphi_1 \wedge \dots \wedge \varphi_n, 0) &:= \Phi(\varphi_1, 0) \vee \dots \vee \Phi(\varphi_n, 0) \\ \Phi(\varphi_1 \rightarrow \varphi_2, 1) &:= (\Phi(\varphi_1, 0) \vee \Phi(\varphi_2, 1)) \wedge (\Phi(\varphi_2, 0) \vee \Phi(\varphi_1, 1)) \\ \Phi(\varphi_1 \rightarrow \varphi_2, 0) &:= (\Phi(\varphi_1, 1) \wedge \Phi(\varphi_2, 0)) \vee (\Phi(\varphi_2, 1) \wedge \Phi(\varphi_1, 0)) \\ \Phi(\text{ite } \varphi_1 \varphi_2 \varphi_3, 1) &:= (\Phi(\varphi_1, 0) \vee \Phi(\varphi_2, 1)) \wedge (\Phi(\varphi_1, 1) \vee \Phi(\varphi_3, 1)) \\ \Phi(\text{ite } \varphi_1 \varphi_2 \varphi_3, 0) &:= (\Phi(\varphi_1, 1) \wedge \Phi(\varphi_2, 0)) \vee (\Phi(\varphi_1, 0) \wedge \Phi(\varphi_3, 0)) \\ \Phi(\forall x_1, \dots, x_n. \varphi, 1) &:= \forall x_1, \dots, x_n. \Phi(\varphi, 1) \\ \Phi(\exists x_1, \dots, x_n. \varphi, 1) &:= \exists x_1, \dots, x_n. \Phi(\varphi, 1) \\ \Phi(\forall x_1, \dots, x_n. \varphi, 0) &:= \exists x_1, \dots, x_n. \Phi(\varphi, 0) \\ \Phi(\exists x_1, \dots, x_n. \varphi, 0) &:= \forall x_1, \dots, x_n. \Phi(\varphi, 0) \\ \Phi(\varphi, 1) &:= \varphi \\ \Phi(\varphi, 0) &:= \neg\varphi \end{aligned}$$

86. subproof

The subproof rule completes a subproof and discharges local assumptions. Every subproof starts with some input steps. The last step of the subproof is the conclusion.

$\triangleright \text{i}_1.$	ψ_1	(input)
\vdots	\vdots	
$\triangleright \text{i}_n.$	ψ_n	(input)
\vdots	\vdots	
$\triangleright \text{j.}$	φ	(\dots)
$\triangleright \text{k.}$	$\neg\psi_1 \dot{\vee} \dots \dot{\vee} \neg\psi_n \dot{\vee} \varphi$	(subproof)

87. bind

The `bind` rule is used to rename bound variables.

$$\frac{\Gamma, y_1, \dots, y_n, x_1 \mapsto y_1, \dots, x_n \mapsto y_n, \triangleright j. \quad \begin{array}{c} \vdots \\ \varphi \leftrightarrow \varphi' \end{array} \quad (\dots)}{\Gamma \triangleright \mathfrak{k}. \quad \forall x_1, \dots, x_n. \varphi \leftrightarrow \forall y_1, \dots, y_n. \varphi' \quad (\text{bind})}$$

where the variables y_1, \dots, y_n is not free in $\forall x_1, \dots, x_n. \varphi$.

88. let

This rule eliminats `let`. It has the form

$$\frac{\Gamma \triangleright i_1. \quad t_1 \simeq s_1 \quad (\dots)}{\Gamma \triangleright i_n. \quad t_n \simeq s_n \quad (\dots)} \quad \begin{array}{c} \vdots \\ u \simeq u' \end{array} \quad (\dots)$$

$$\frac{\Gamma, x_1 \mapsto s_1, \dots, x_n \mapsto s_n, \triangleright j. \quad \begin{array}{c} \vdots \\ u \simeq u' \end{array} \quad (\dots)}{\Gamma \triangleright \mathfrak{k}. \quad (\text{let } x_1 := t_1, \dots, x_n := t_n. u) \simeq u' \quad (\text{let}; i_1 \dots i_n)}$$

where \simeq is replaced by \leftrightarrow where necessary.

If for $t_i \simeq s_i$ the t_i and s_i are syntactically equal, the premise is skipped.

89. onepoint

The `onepoint` rule is the “one-point-rule”. That is: it eliminates quantified variables that can only have one value.

$$\frac{\Gamma, x_{k_1}, \dots, x_{k_m}, x_{j_1} \mapsto t_{j_1}, \dots, x_{j_o} \mapsto t_{j_o}, \triangleright j. \quad \begin{array}{c} \vdots \\ \varphi \leftrightarrow \varphi' \end{array} \quad (\dots)}{\Gamma \triangleright \mathfrak{k}. \quad Qx_1, \dots, x_n. \varphi \leftrightarrow Qx_{k_1}, \dots, x_{k_m}. \varphi' \quad (\text{onepoint})}$$

where $Q \in \{\forall, \exists\}$, $n = m + o$, k_1, \dots, k_m and j_1, \dots, j_o are monotone mappings to $1, \dots, n$, and no x_{k_i} appears in x_{j_1}, \dots, x_{j_o} .

The terms t_{j_1}, \dots, t_{j_o} are the points of the variables x_{j_1}, \dots, x_{j_o} . Points are defined by equalities $x_i \simeq t_i$ with positive polarity in the term φ .

Remark. Since an eliminated variable x_i might appear free in a term t_j , it is necessary to replace x_i with t_i inside t_j . While this substitution is performed correctly, the proof for it is currently missing.

Example 89.1. An appliction of the `onepoint` rule on the term $\forall x, y. x \simeq y \rightarrow f(x) \wedge f(y)$ look like this:

```
(anchor :step t3 :args ((:= y x)))
(step t3.t1 (cl (= x y)) :rule refl)
```

```

(step t3.t2 (cl (= (= x y) (= x x)))
  :rule cong :premises (t3.t1))
(step t3.t3 (cl (= x y)) :rule refl)
(step t3.t4 (cl (= (f y) (f x)))
  :rule cong :premises (t3.t3))
(step t3.t5 (cl (= (and (f x) (f y)) (and (f x) (f x))))
  :rule cong :premises (t3.t4))
(step t3.t6 (cl (=
  (=> (= x y) (and (f x) (f y)))
  (=> (= x x) (and (f x) (f x)))))
  :rule cong :premises (t3.t2 t3.t5))
(step t3 (cl (=
  (forall ((x S) (y S)) (=> (= x y) (and (f x) (f y))))
  (forall ((x S)) (=> (= x x) (and (f x) (f x)))))
  :rule qnt_simplify)

```

90. sko_ex

The sko_ex rule skolemizes existential quantifiers.

$$\frac{\Gamma, x_1 \mapsto (\epsilon x_1.\varphi), \dots, x_n \mapsto (\epsilon x_n.\varphi), \triangleright j. \quad \begin{array}{c} \vdots \\ \varphi \leftrightarrow \psi \end{array} \quad (\dots)}{\Gamma \triangleright \mathfrak{k}. \quad \exists x_1, \dots, x_n. \varphi \leftrightarrow \psi \quad (\text{sko_ex})}$$

91. sko_forall

The sko_forall rule skolemizes universal quantifiers.

$$\frac{\Gamma, x_1 \mapsto (\epsilon x_1.\neg\varphi), \dots, x_n \mapsto (\epsilon x_n.\neg\varphi), \triangleright j. \quad \begin{array}{c} \vdots \\ \varphi \leftrightarrow \psi \end{array} \quad (\dots)}{\Gamma \triangleright \mathfrak{k}. \quad \forall x_1, \dots, x_n. \varphi \leftrightarrow \psi \quad (\text{sko_forall})}$$

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